

Fundamental thm of group homo.
The first theorem of isomorphism.

Statement: - If $f: G \rightarrow G'$ is a homomorphism.

$\therefore f: G \rightarrow G'$ $G' = \{f(a) / a \in G\}$ homomorphic image of group G

If $K = \ker f$ then $\frac{G}{K} \cong G'$

(OR)

every homomorphic image of a group G is isomorphic to some quotient group of G .

$\therefore \frac{G}{\frac{K}{\ker f}} \cong G' \quad (i, \ker f \triangleleft G)$

Prf. Define a mapping $\phi: \frac{G}{K} \rightarrow G'$

such that $\phi(ka) = f(a)$ ($\because \frac{G}{K} = \{ka / a \in G, k \in \ker f\}$)

To S.T ϕ is isomorphism

First S.T ϕ is well-defined

Let $ka = kb, ka, kb \in \frac{G}{K}$

$\Rightarrow ab^{-1} \in K = \ker f$

$\Rightarrow f(ab^{-1}) = e'$

$\Rightarrow f(a)[f(b)]^{-1} = e'$

$\Rightarrow f(a) = e' f(b)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \phi(ka) = \phi(kb)$

$\therefore \phi$ is well-defined.

$\therefore \phi$ is well-defined (1-1)

ϕ is onto $\phi: \frac{G}{K} \rightarrow G'$

Let $g' \in G'$

Since $f: G \rightarrow G'$ is onto

$\therefore \exists g \in G$ such that $f(g) = g'$
 $\exists kg \in \frac{G}{K}$ such that $\phi(kg) = f(g) = g'$

$\therefore \phi(kg) = g', g'$ is arbitrary

$\therefore \forall kg \in \frac{G}{K} \exists g \in G$ such that $\phi(kg) = g'$
 $\therefore \phi$ is onto

ϕ is homo:-

$$\begin{aligned}
 \phi(ka kb) &= \phi(kab) & (\because G \text{ is Factor} \\
 &= f(ab) & (ka \cdot kb = kab \\
 &= f(a)f(b) & \phi|_K = \ker f) \\
 &= \phi(ka) \phi(kb) & (\because f \text{ is Hom}
 \end{aligned}$$

$$\therefore \phi(ka kb) = \phi(ka) \phi(kb)$$

$\therefore \phi$ is homo

$\therefore \phi$ is isomorphism ($\because \phi$ is one-one onto \Rightarrow bijective)

$$G/K \cong \frac{G}{\ker f}$$

$\therefore \exists$ an isomorphism from $\frac{G}{K}$ to G' such that

$$\therefore \frac{G}{K} \cong G'$$